Chapter 4

Differential Analysis of Fluid Flow

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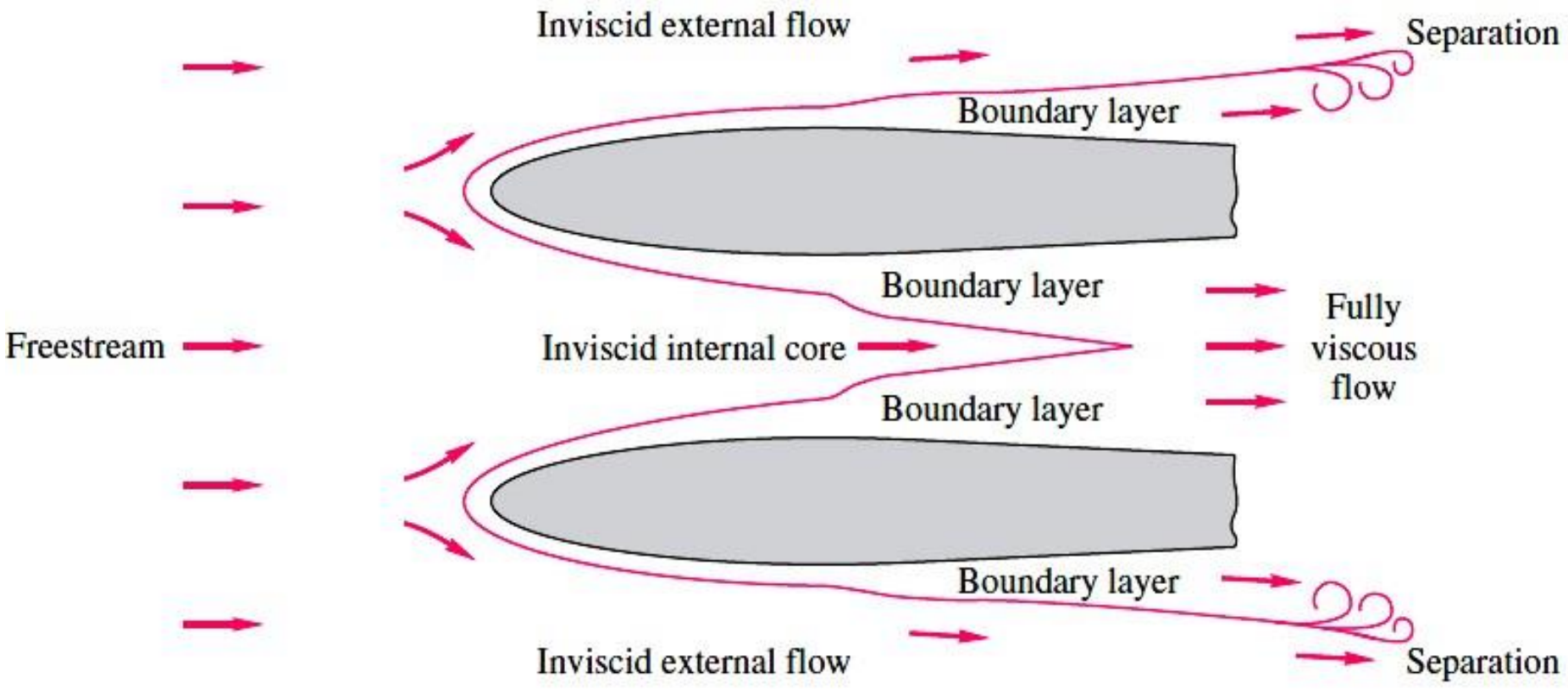
Introduction

- All real fluids possess viscosity, and are, to some degree, compressible.
- Nevertheless there are many instances in which the behavior of real fluids quite closely approaches that of an inviscid fluid.
- It has already been remarked that the flow of a real fluid may frequently be subdivided into two regions: one, adjacent to the solid boundaries of the flow, is a thin layer in which **viscosity has a considerable effect**; in the other region, constituting the remainder of the flow, the **viscous effects are negligible.**

Introduction

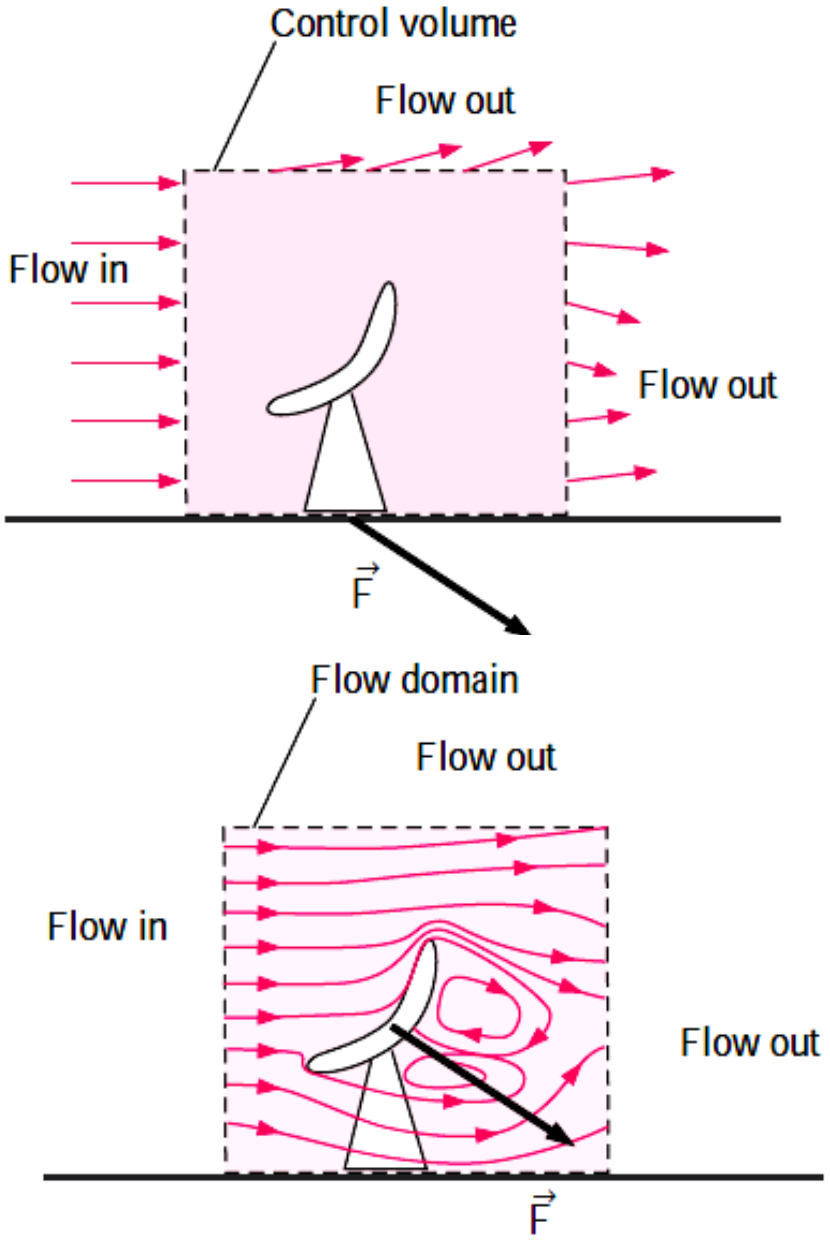
- Most CFD (Computational Fluid Dynamics) solvers use potential flow theory to approximate the flow solution outside the boundary layers in incompressible flows



Differential Analysis of Fluid Flow

- In analyzing fluid motion, we might take one of two paths:

1. Seeking an estimate of gross effects (mass flow, induced force, energy change) over a finite region or control volume or
2. Seeking the point-by-point details of a flow pattern by analyzing an infinitesimal region of the flow.



Differential Analysis of Fluid Flow

- The control volume technique is useful when we are interested in the overall features of a flow, such as mass flow rate into and out of the control volume or net forces applied to bodies.
- Differential analysis, on the other hand, involves application of differential equations of fluid motion to *any and every point in the flow field over a region* called the flow domain.
- When solved, these differential equations yield details about the velocity, density, pressure, etc., at *every point* throughout the *entire flow domain.*

The Acceleration Field of a Fluid

- Velocity is a vector function of position and time and thus has three components *u*, *v*, and *w*, each a scalar field in itself.

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}v(x, y, z, t) + \mathbf{k}w(x, y, z, t)$$

- This is the most important variable in fluid mechanics: Knowledge of the velocity vector field is nearly equivalent to *solving a fluid flow problem.*
- The acceleration vector field a of the flow is derived from Newton's second law by computing the total time derivative of the velocity vector:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt}$$

The Acceleration Field of a Fluid

- Since each scalar component (u, v , w) is a function of the four variables (x, y, z, t), we use the chain rule to obtain each scalar time derivative. For example,

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

- But, by definition, *dx/dt* is the local velocity component *u*, *and dy/dt =v , and dz/dt = w.*
- The total time derivative of *u* may thus be written as follows, with exactly similar expressions for the time derivatives of *v and w:*

The Acceleration Field of a Fluid

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V}\cdot\nabla)v$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w$$

- Summing these into a vector, we obtain the total

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}$$

The Acceleration Field of a Fluid

- The term $\delta V/\delta t$ is called the **local acceleration**, which vanishes if the flow is steady-that is, independent of time.
- The three terms in parentheses are called the **convective acceleration**, which arises when the particle moves through regions of spatially varying velocity, as in a nozzle or diffuser.
- The gradient operator ∇ is given by:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \mathbf{V}\cdot\nabla$$

The Acceleration Field of a Fluid

- The total time derivative—sometimes called the ***substantial or material derivative***— concept may be applied to any variable, such as the pressure:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V}\cdot\nabla)p$$

- Wherever convective effects occur in the basic laws involving mass, momentum, or energy, the basic differential equations become nonlinear and are usually more complicated than flows that do not involve convective changes.

Example

Given the eulerian velocity vector field

$$\mathbf{V} = 3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}$$

find the total acceleration of a particle.

Solution

- *Assumptions:* Given three known unsteady velocity components, $u = 3t$, $v = xz$, and $w = ty^2$.

- *Solution step 1:* First work out the local acceleration $\partial\mathbf{V}/\partial t$:

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{i}\frac{\partial u}{\partial t} + \mathbf{j}\frac{\partial v}{\partial t} + \mathbf{k}\frac{\partial w}{\partial t} = \mathbf{i}\frac{\partial}{\partial t}(3t) + \mathbf{j}\frac{\partial}{\partial t}(xz) + \mathbf{k}\frac{\partial}{\partial t}(ty^2) = 3\mathbf{i} + 0\mathbf{j} + y^2\,\mathbf{k}$$

Solution step 2: In a similar manner, the convective acceleration terms, are

Solution step 2: In a similar manner, the convective acceleration terms, are

$$u\frac{\partial \mathbf{V}}{\partial x} = (3t)\frac{\partial}{\partial x}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (3t)(0\mathbf{i} + z\mathbf{j} + 0\mathbf{k}) = 3tz\,\mathbf{j}$$

$$v\frac{\partial \mathbf{V}}{\partial y} = (xz)\frac{\partial}{\partial y}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (xz)(0\mathbf{i} + 0\mathbf{j} + 2ty\mathbf{k}) = 2txyz\,\mathbf{k}$$

$$w\frac{\partial \mathbf{V}}{\partial z} = (ty^2)\frac{\partial}{\partial z}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (ty^2)(0\mathbf{i} + x\mathbf{j} + 0\mathbf{k}) = txy^2\,\mathbf{j}$$

- *Solution step 3:* Combine all four terms above into the single "total" or "substantial" derivative:

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = (3\mathbf{i} + y^2\mathbf{k}) + 3tz\mathbf{j} + 2txyz\mathbf{k} + txy^2\mathbf{j}$$

$$= 3\mathbf{i} + (3tz + txy^2)\mathbf{j} + (y^2 + 2txyz)\mathbf{k} \qquad \textit{Ans.}$$

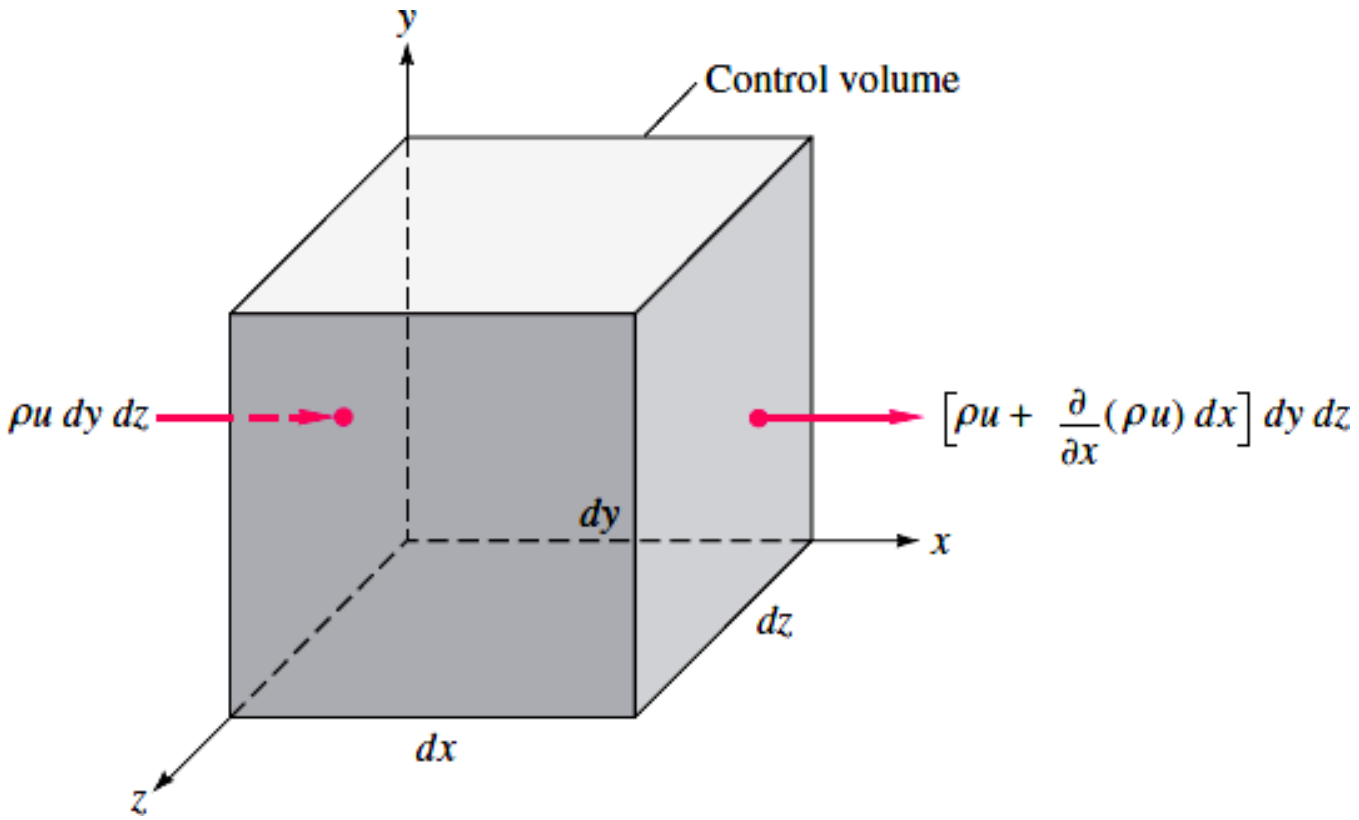
- *Comments:* Assuming that **V** is valid everywhere as given, this total acceleration vector $d\mathbf{V}/dt$ applies to all positions and times within the flow field.

The Differential Equation of Mass Conservation

- Conservation of mass, often called the ***continuity relation***, states that the fluid mass cannot change.
- We apply this concept to a very small region. All the basic differential equations can be derived by considering either an elemental control volume or an elemental system.
- We choose an infinitesimal fixed control volume (*dx*, *dy*, *dz)*, as in shown in fig below, and use basic control volume relations.
- The flow through each side of the element is approximately one-dimensional, and so the appropriate mass conservation relation to use here is

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

The Differential Equation of Mass Conservation



- The element is so small that the volume integral simply reduces to a differential term:

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dx\, dy\, dz$$

The Differential Equation of Mass Conservation

- The mass flow terms occur on all six faces, three inlets and three outlets.
- Using the field or continuum concept where all fluid properties are considered to be uniformly varying functions of time and position, such as $\rho = \rho\,(x, y, z, t)$.
- Thus, if T is the temperature on the left face of the element, the right face will have a slightly different temperature $T + (\partial T/\partial x)\,dx$.
- For mass conservation, if ρu *is* known on the left face, the value of this product on the right face is $\rho u + (\partial \rho u/\partial x)\,dx$.

The Differential Equation of Mass Conservation

Face	Inlet mass flw	Outlet mass flw
x	$\rho u\, dy\, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz$
y	$\rho v\, dx\, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\, dy\right] dx\, dz$
z	$\rho w\, dx\, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx\, dy$

- Introducing these terms into the main relation

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz + \frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz + \frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz = 0$$

- Simplifying gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

equation of continuity

The Differential Equation of Mass Conservation

- The vector gradient operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

- enables us to rewrite the equation of continuity in a compact form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \equiv \nabla \cdot (\rho \mathbf{V})$$

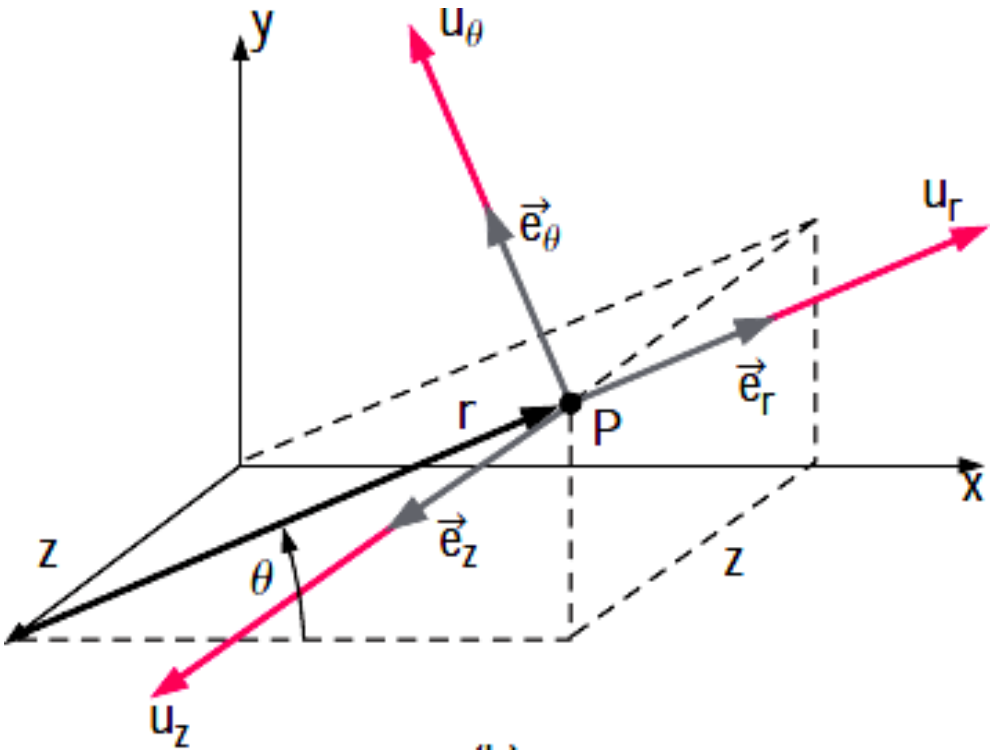
- so that the compact form of the continuity relation is

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0}$$

The Differential Equation of Mass Conservation

Continuity Equation in Cylindrical Coordinates

- Many problems in fluid mechanics are more conveniently solved in cylindrical coordinates (r, θ, z) (often called cylindrical polar coordinates), rather than in Cartesian coordinates.



The Differential Equation of Mass Conservation

- Continuity equation in cylindrical coordinates is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Steady Compressible Flow

- $\partial/\partial t \equiv 0$ ow is steady ,

and all properties are functions of position only and the continuity equation reduces to

Cartesian:
$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical:
$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

The Differential Equation of Mass Conservation

Incompressible Flow

- A special case that affords great simplification is incompressible flow, where the density changes are negligible. Then $\partial\rho/\partial t \approx 0$ regardless of whether the flow is steady or unsteady,
- The result

$$\nabla \cdot \mathbf{V} = 0$$

- is valid for steady or unsteady incompressible flow. The two coordinate forms are

Cartesian:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

The Differential Equation of Mass Conservation

- The criterion for incompressible flow is $\mathrm{Ma} \leq 0.3$
- where Ma = V/a is the dimensionless Mach number of the flow.
- For air at standard conditions, a flow can thus be considered incompressible if the velocity is less than about 100 m/s.

The Differential Equation of Linear Momentum

- Using the same elemental control volume as in mass conservation, for which the appropriate form of the linear momentum relation is

$$\sum \mathbf{F} = \frac{\partial}{\partial t}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V} \right) + \sum (\dot{m}_i \mathbf{V}_i)_{\text{out}} - \sum (\dot{m}_i \mathbf{V}_i)_{\text{in}}$$

- Again the element is so small that the volume integral simply reduces to a derivative term:

$$\frac{\partial}{\partial t}(\mathbf{V}\rho \, d\mathcal{V}) \approx \frac{\partial}{\partial t}(\rho \mathbf{V}) \, dx \, dy \, dz$$

- The momentum fluxes occur on all six faces, three inlets and three outlets.

The Differential Equation of Linear Momentum

Faces	Inlet momentum flx	Outlet momentum flx
x	$\rho u\mathbf{V}\ dy\ dz$	$\left[\rho u\mathbf{V} + \frac{\partial}{\partial x}(\rho u\mathbf{V})\ dx\right] dy\ dz$
y	$\rho v\mathbf{V}\ dx\ dz$	$\left[\rho v\mathbf{V} + \frac{\partial}{\partial y}(\rho v\mathbf{V})\ dy\right] dx\ dz$
z	$\rho w\mathbf{V}\ dx\ dy$	$\left[\rho w\mathbf{V} + \frac{\partial}{\partial z}(\rho w\mathbf{V})\ dz\right] dx\ dy$

- Introducing these terms

$$\sum \mathbf{F} = dx\ dy\ dz\left[\frac{\partial}{\partial t}(\rho\mathbf{V}) + \frac{\partial}{\partial x}(\rho u\mathbf{V}) + \frac{\partial}{\partial y}(\rho v\mathbf{V}) + \frac{\partial}{\partial z}(\rho w\mathbf{V})\right]$$

The Differential Equation of Linear Momentum

- A simplification occurs if we split up the term in brackets as follows:

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho v \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V})$$

$$= \rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)$$

- The long term in parentheses on the right-hand side is the total acceleration of a particle that instantaneously occupies the control volume:

$$\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{d\mathbf{V}}{dt}$$

The Differential Equation of Linear Momentum

- Thus we have now we have

$$\sum \mathbf{F} = \rho \frac{d\mathbf{V}}{dt} dx\, dy\, dz$$

- This equation points out that the net force on the control volume must be of differential size and proportional to the element volume.
- These forces are of two types, ***body forces and surface forces.*** Body forces are due to external fields (gravity, magnetism, electric potential) that act on the entire mass within the element. The only body force we shall consider is gravity.
- The gravity force on the differential mass $\rho\, dx\, dy\, dz$ within the control volume is

$$d\mathbf{F}_{\text{grav}} = \rho \mathbf{g}\, dx\, dy\, dz$$

Newtonian Fluid: Navier-Stokes Equations

- For a newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity.

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

- where μ *is the viscosity coefficient*
- Substitution gives the differential momentum equation for a newtonian fluid with constant density and viscosity:

Differential momentum equation in its full glory:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

The differential momentum equation for a Newtonian fluid with constant density and viscosity:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\frac{dw}{dt}$$

Navier-Stokes equations

Inviscid Flow

- Shearing stresses develop in a moving fluid because of the viscosity of the fluid.
- We know that for some common fluids, such as air and water, the viscosity is small, therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity (and thus shearing stresses).
- Flow fields in which the shearing stresses are assumed to be negligible are said to be ***inviscid, nonviscous, or frictionless.***
- For fluids in which there are no shearing stresses the normal stress at a point is independent of direction—that is $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$.

Inviscid Flow

Euler's Equations of Motion

- For an inviscid flow in which all the shearing stresses are zero and the Euler's equation of motion is written as1q

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

- In vector notation Euler's equations can be expressed as

$$\rho\mathbf{g} - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]$$

MILLENIUM PRIZE PROBLEMS

1. **Riemann Hypothesis**
   Number Theory

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

2. **Birch & Swinnerton Dyer Conjecture**
   Number theory

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

3. **P versus NP problem**
   Computer Science

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

4. **Navier-Stokes existence & smoothness**
   Partial Differential equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

5. **Hodge Conjecture**
   Algebraic Geometry

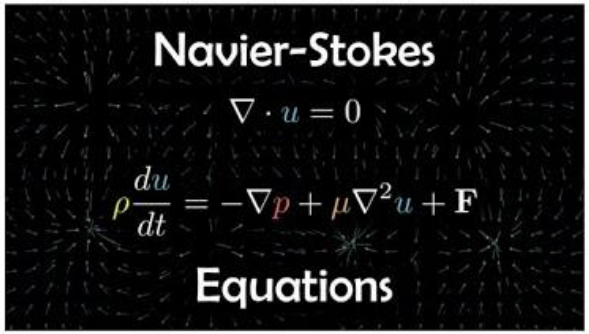
The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

6. **Yang-Mills existence & mass gap**
   Quantum Physics

Experiment and computer simulations suggest the existence of a "mass gap" in

Today there is actually 1M prize problem for the Navier-Stokes equation!

Millenium Prize Problems



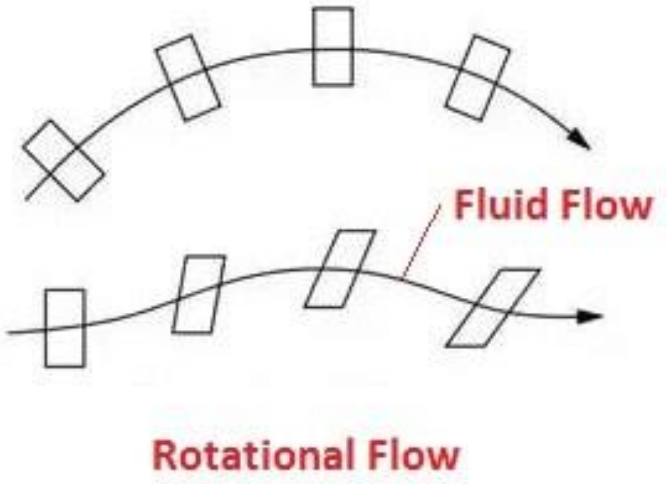
Navier-Stokes Existence and Smoothness

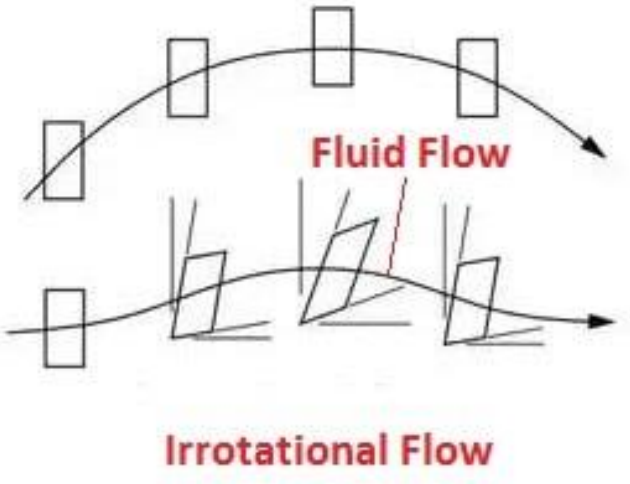
Rotationality, Irrotationality and Vorticity

Rotationality: refers to the actual **physical spinning** of a fluid particle about its own center of mass.
Examples: Tornadoes, whirlpools, and turbulent flow in pipes.

Irrotationality: The entire flow **has zero rotation** at every point. Example: Idealized flow around airfoils, free-stream flow in aerodynamics, and certain flows invicid liquids fluids

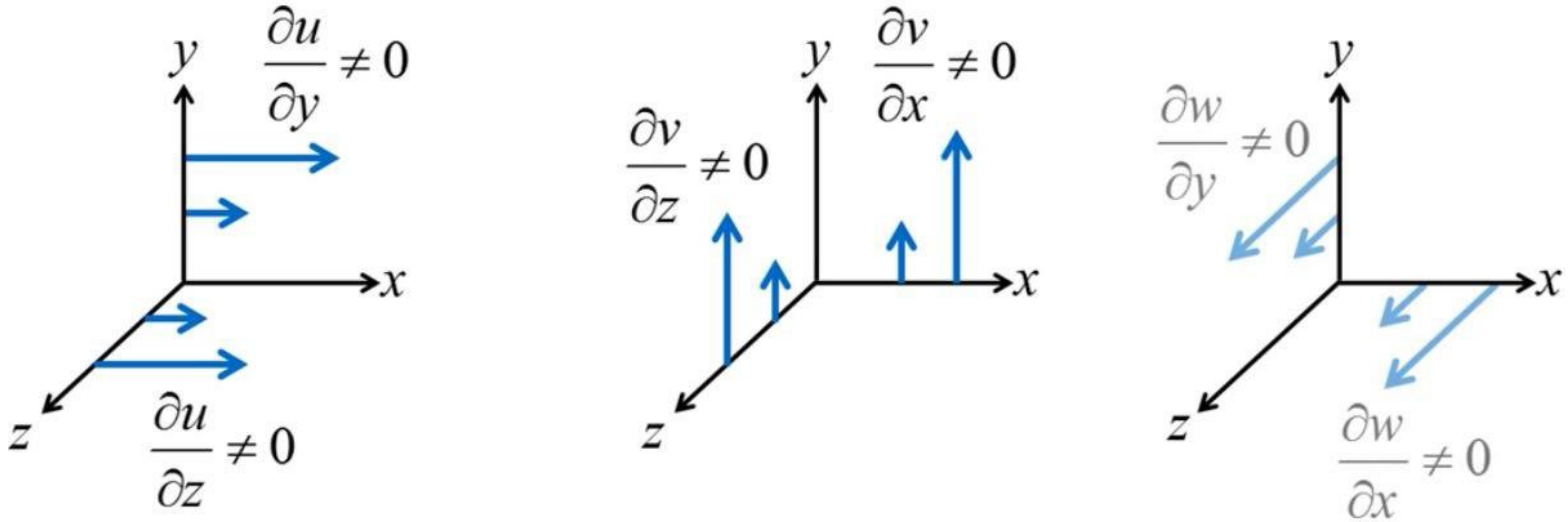
Vorticity: measures the local **tendency of fluid to rotate**, based on velocity gradients. It is the measure of the angular velocity that's casing rotation.





Vorticity and Irrotationality

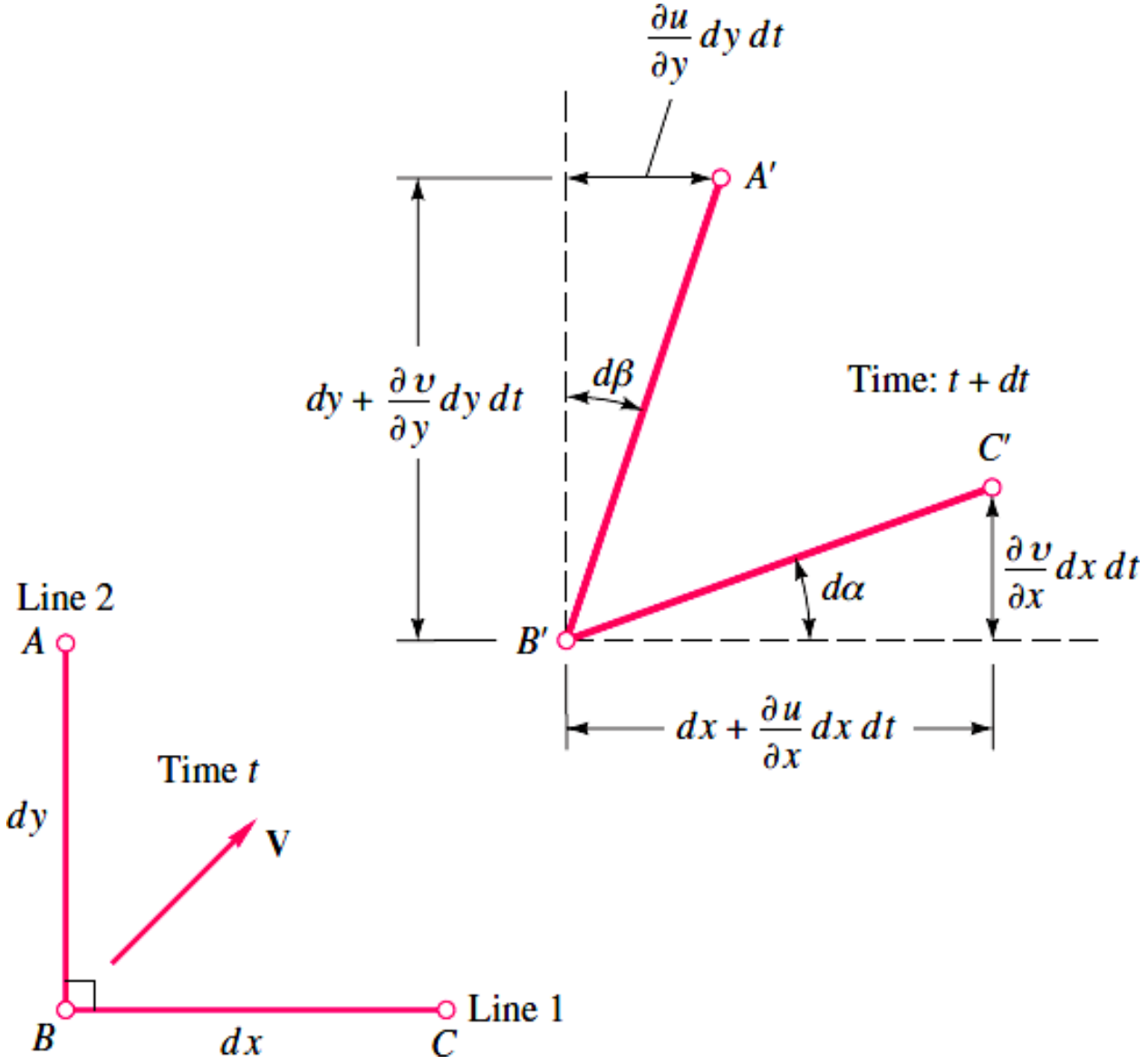
- The assumption of zero fluid angular velocity, or irrotationality, is a very useful simplification.
- Here we show that angular velocity is associated with the curl of the local velocity vector.
- The rotation (angular velocity) occurs if there is a velocity gradient other than the direction of motion.



Vorticity and Irrotationality

- The differential relations for deformation of a fluid element can be derived by examining the Fig. below.
- Two fluid lines AB and BC, initially perpendicular at time t, move and deform so that at t + dt they have slightly different lengths A’B’ and B’C’ and are slightly off the perpendicular by angles d and d.

Vorticity and Irrotationality



Vorticity and Irrotationality

- We define the angular velocity ω_z about the z axis as the average rate of counterclockwise turning of the two lines:

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$$

- But from the fig. *dα and dβ* are each directly related to velocity derivatives in the limit of small dt:

$$d\alpha = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial v/\partial x)\,dx\,dt}{dx + (\partial u/\partial x)\,dx\,dt}\right] = \frac{\partial v}{\partial x}dt$$

$$d\beta = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial u/\partial y)\,dy\,dt}{dy + (\partial v/\partial y)\,dy\,dt}\right] = \frac{\partial u}{\partial y}dt$$

- Substitution results

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Vorticity and Irrotationality

In exactly similar manner we determine the other two rates:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

- The vector $\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$ is thus one-half the curl of the velocity vector

$$\boldsymbol{\omega} = \frac{1}{2}(\text{curl } \mathbf{V}) = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

- A vector twice as large is called the ***vorticity***

$$\boldsymbol{\zeta} = 2\boldsymbol{\omega} = \text{curl } \mathbf{V}$$

Vorticity and Irrotationality

- Many flows have negligible or zero vorticity and are called ***irrotational.***

$$\text{curl } \mathbf{V} \equiv 0$$

- **Example.** For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = (x^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$$

- Is this flow irrotational?

Solution.

- For the prescribed velocity field

$$u = x^2 - y^2 \qquad v = -2xy \qquad w = 0$$

Vorticity and Irrotationality

and therefore

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[(-2y) - (-2y)\right] = 0$$

Thus, the flow is irrotational.

The Stream Function

- Steady, incompressible, plane, two-dimensional flow represents one of the simplest types of flow of practical importance.
- By plane, two-dimensional flow we mean that there are only two velocity components, such as *u* and *v* when the flow is considered to be in the x–y plane.
- For this flow the continuity equation, reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- The stream function ψ is a clever device that allows us to satisfy the continuity equation and then solve the momentum equation directly for the single variable . Lines of constant ψ are streamlines of the flow.

The Stream Function

- This equation is satisfied identically if a function ψ(x, y) is defined such that the continuity equation becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right)=\frac{\partial^2 \psi}{\partial x\,\partial y}-\frac{\partial^2 \psi}{\partial y\,\partial x}=0$$

- This shows that this new function ψ must be defined such that

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$

- or

$$\mathbf{V}=\mathbf{i}\frac{\partial \psi}{\partial y}-\mathbf{j}\frac{\partial \psi}{\partial x}$$

The Stream Function

- The stream function simplifies the analysis by having to determine only one unknown function, $\psi(x, y)$ rather than the two functions $u(x, y)$ and $v(x, y)$.
- Another particular advantage of using the stream function is related to the fact that lines along which ψ is constant are streamlines.
- Streamlines are lines in the flow field that are everywhere tangent to the velocities. It follows from the definition of the streamline that the slope at any point along a streamline is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

The Stream Function

- The change in the value of ψ as we move from one point (x, y) to a nearby point $(x + dx, y + dy)$ is given by the relationship:

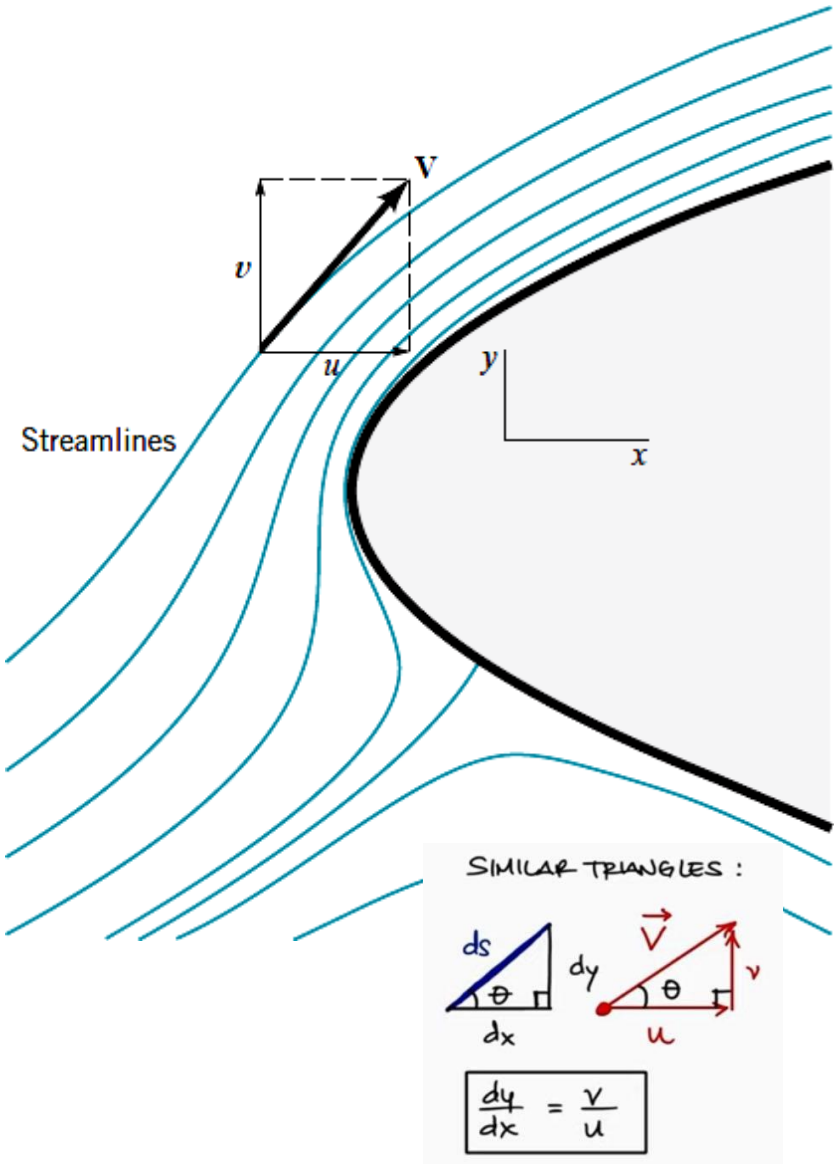
$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\, dx + u\, dy$$

- Along a line of constant ψ we have $d\psi = 0$ so that

$$-v\, dx + u\, dy = 0$$

- and, therefore, along a line of constant ψ

$$\frac{dy}{dx} = \frac{v}{u}$$

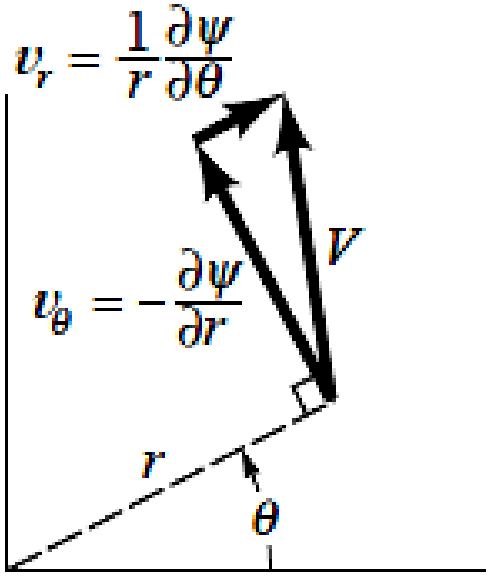


The Stream Function

- In cylindrical coordinates the continuity equation for incompressible, plane, two dimensional flow reduces to

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

- and the velocity components, v_r and v_θ can be related to the stream function, $\psi(r, \theta)$ through the equations



$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

Example: Stream function

Find and interpret if there exists a stream function for the velocity field below:

$$u = a(x^2 - y^2) \quad v = -2axy \quad w = 0$$

Solution: If the flow field satisfies the continuity equation we can be sure that the stream function does exists. Lets check:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Substitute: $2ax + (-2ax) = 0$ checks

Therefore we are certain that a stream function exists. To find ψ, we simply set

$$u = \frac{\partial \psi}{\partial y} = ax^2 - ay^2 \qquad (1)$$

$$v = -\frac{\partial \psi}{\partial x} = -2axy \qquad (2)$$

and work from either one toward the other. Integrate (1) partially

$$\psi = ax^2y - \frac{ay^3}{3} + f(x)$$

Differentiate (3) with respect to x and compare with (2)

$$\frac{\partial \psi}{\partial x} = 2axy + f'(x) = 2axy$$

Therefore $f'(x) = 0$, or $f =$ constant. The complete stream function is thus found

$$\psi = a\left(x^2y - \frac{y^3}{3}\right) + C$$

Examples: Conservation of mass

An incompressible flow has the velocity components:

$$u = x^3 + 2z^2$$

$$w = y^3 - 2yz$$

Determine the most general form of v (the y component of **v**) which satisfies the conservation of mass:

Solution:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}$$

$$= -(-2 \cdot y) - 3x^2$$

$$\frac{\partial v}{\partial y} = 2 \cdot y - 3x^2$$

$$\int \partial v = \int 2 \cdot y \, \partial y + - \int 3 \cdot x^2 \cdot \partial y$$

$$= y^2 - 3 \cdot x^2 \cdot y$$

$$\boxed{v = y^2 - 3 \cdot x^2 \cdot y + f(x, z)}$$

End of Chapter 4